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Linear-quadratic control and quadratic differential forms for multidimensional behaviors

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ABSTRACT

This paper deals with systems described by constant coefficient linear partial differential equations (nD -systems) from a behavioral point of view. In this context we treat the linear-quadratic control problem where the performance functional is the integral of a quadratic differential form. We look for characterizations of the set of stationary trajectories and of the set of local minimal trajectories with respect to compact support variations, turning out that they are equal if the system is dissipative. Finally we provide conditions for regular implementability of this set of trajectories and give an explicit representation of an optimal controller.

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1. Introduction

Linear-quadratic (LQ) optimal control was initially developed for finite dimensional input-state-output systems [1,6], with performance functional given by the integral of a quadratic function of the input and the state, in one independent variable (usually time). More concretely, the classical linear-quadratic optimal control problem (LQ problem) is formulated as follows: given a finite dimensional linear time invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad (1)$$

and a given initial state $x(0) = x_0$, find a control input function u such that the cost functional

$$\int_0^\infty u^\top(t)Ru(t) + x^\top(t)Lx(t)dt \quad (2)$$

is minimized.

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Many continuous-time dynamical models of physical systems involve both time *and* space variables, and are hence given by partial differential equations. An important and generally appreciated approach to linear-quadratic optimal control in the context of systems described by linear partial differential equations (also called nD systems) has been to represent the partial differential equations in input-state-output form using infinite dimensional state spaces and the theory of semi groups, see e.g. [3].

Often, the nD systems under consideration do not have a clear input-state-output structure. In addition, the cost functional may involve higher order derivatives of the system variables. To deal with this situation while avoiding infinite dimensional state space theory, an approach to linear-quadratic optimal control based on the theory of differential modules was developed for a large class of controllable nD systems in [15] and later on extended to non-controllable nD systems in [17].

In this paper we aim to provide a formulation of the linear-quadratic problem that deals with systems that are not necessarily in state space form, in which there is no a priori input/output partition of the system variables, and whose dynamics is described by linear constant-coefficient partial differential equations. Furthermore, the cost functional is allowed to be the integral of an arbitrary quadratic expression in the system variables *and their higher order derivatives*.

We show that the behavioral approach to systems and control, initially developed by J.C. Willems (see [14]), provides an elegant and efficient framework for dealing with such problem. In the behavioral context the problem considered in this paper can be stated as follows: given a plant and a quadratic differential form (in the following abbreviated with QDF), characterize the trajectories of the plant that are stationary or optimal with respect to the integral of the QDF, and investigate the existence and representation of optimal *regular* controllers. In the context of 1D systems, this linear-quadratic control problem has been treated before in [26]. In the context of nD systems some previous work was done considering optimal trajectories of plants that do not impose conditions on the trajectories, i.e., where the trajectories are free in the plant, see [12,19].

We emphasize that we furnish practical techniques for the solvability of the problems stated and even though the results and computations in this paper are matrix based and therefore representation dependent, the characterizations derived are not.

A preliminary version of this paper has been presented in [7]. The outline of the present paper is as follows: we begin by introducing some background material on multidimensional (nD) behavioral theory. Most of this material is standard, centering around concepts such as kernel representation, orthogonal module and latent variable representation. In Section 3 we review the classical properties of controllability and observability and present some relations of these properties with trajectories of compact support. We also introduce the notion of *faithful* image representation. Section 4 is devoted to an exposition of quadratic differential forms and the notion of system dissipativity. In Section 5 we find an explicit representation of the set of stationary trajectories. We prove that this set is equal to the set of local minimum trajectories if the system is dissipative, and empty otherwise. We conclude this section by providing a criterion for checking whether the system is dissipative or not. Finally, in Section 6, the so called *synthesis problem* is addressed, i.e., the problem of finding an nD system, called a controller, that constrains (through a *regular* interconnection) the plant behavior in order to implement the optimal trajectories. A representation of such a controller is found.

2. Multidimensional systems

In behavioral system theory, a *behavior* is a subset of the space $\mathbb{W}^{\mathbb{T}}$ of all functions from \mathbb{T} , the *indexing set*, to \mathbb{W} , the *signal space*. In this paper we consider systems with $\mathbb{T} = \mathbb{R}^n$ (from which the terminology “ nD -system” derives) and $\mathbb{W} = \mathbb{R}^w$. We call \mathfrak{B} a *linear differential nD behavior* or just a *behavior* if it is the space of solutions of a system of linear, constant-coefficient partial differential equations (LPDE); more precisely, if \mathfrak{B} is the subspace of $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^w)$ (the space of all \mathcal{C}^{∞} -functions from \mathbb{R}^n to \mathbb{R}^w) consisting of all solutions w of

$$R\left(\frac{d}{dx}\right)w = 0 \quad (3)$$

where $R(\xi_1, \dots, \xi_n)$ is a polynomial matrix in n indeterminates $\xi_i, i = 1, \dots, n$, and $\frac{d}{dx} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$. We call (3) a *kernel representation* of \mathfrak{B} and write $\mathfrak{B} = \ker \left(R \left(\frac{d}{dx} \right) \right)$. The variable w has w components, it is often called the *external variable*. We denote the set consisting of all linear differential nD -behaviors with w external variables by \mathcal{L}_n^w .

The family of systems \mathcal{L}_n^w enjoys many important properties (see [30,9]). One of this properties is that a behavior $\mathfrak{B} \in \mathcal{L}_n^w$ is uniquely determined by its module of annihilators (also called its orthogonal module), defined by

$$\mathfrak{B}^\perp = \left\{ q \in \mathbb{R}^{1 \times w}[\xi_1, \dots, \xi_n] \mid q \left(\frac{d}{dx} \right) w = 0 \text{ for all } w \in \mathfrak{B} \right\}.$$

The relation between \mathfrak{B} and \mathfrak{B}^\perp is very useful since it establishes an association between algebraic objects on the one hand and the space of trajectories of dynamical systems on the other.

Another important feature is that we can apply the *elimination theorem*. Given a behavior \mathfrak{B} , the elimination theorem states that the projection of \mathfrak{B} onto any subset of its components is also a behavior, i.e., a solution space of a system of LPDE. This is important since, often, the specification of a behavior involves additional, auxiliary variables, called *latent variables* (for example in order to express basic laws involving for instance *internal voltages* and *currents* in electrical circuits in order to express the external port behavior). For given polynomial matrices R and M in n indeterminates $\xi_i, i = 1, \dots, n$, the elimination theorem states that the subspace of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ consisting of all functions w for which there exists $\ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ such that

$$R \left(\frac{d}{dx} \right) w = M \left(\frac{d}{dx} \right) \ell \quad (4)$$

is again a linear differential nD -system, i.e., there exists a polynomial matrix R' in n indeterminates such that

$$\mathfrak{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^l) \text{ such that (4) holds} \}$$

is equal to $\ker \left(R' \left(\frac{d}{dx} \right) \right)$. We call (4) a *latent variable representation* of \mathfrak{B} and the variable ℓ is called the *latent variable*. The external variable w , the variable whose behavior the model aims at, is often called the *manifest variable*.

Remark 2.1. The assumption that the underlying function space is equal to $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ is crucial. For example, if we restrict ourselves to \mathcal{C}^∞ solutions with *compact support*, then the one-to-one correspondence between \mathfrak{B} and its module of annihilators breaks down, and the elimination theorem will no longer hold (see [22]).

3. Controllability, observability and faithfulness

In this section we review the properties of controllability and observability. These properties were initially defined for 1D systems in the behavioral context in [27] and naturally generalized to nD systems in [11]. Here, we pay special attention to the relation of these properties with trajectories of \mathfrak{B} of compact support.

Definition 3.1. A behavior $\mathfrak{B} \in \mathcal{L}_n^w$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ and all subsets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closure, there exist $w \in \mathfrak{B}$ such that $w_1 = w|_{U_1}$ and $w_2 = w|_{U_2}$.

The above definition means that for any pair of trajectories w_1 and w_2 in the behavior there exists a trajectory w in the behavior that coincides with w_1 on U_1 and with w_2 on U_2 . Intuitively, w has patched up w_1 and w_2 .

There are a number of characterizations of controllability but the one useful for our purposes is the equivalence of controllability with the existence of an image representation. Consider the following special latent variable representation:

$$w = M \left(\frac{d}{dx} \right) \ell$$

with $M \in \mathbb{R}^{w \times \ell}[\xi]$. Such special latent variable representations often appear in physics, where the latent variables in a such representation are called *potentials*. Clearly, $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dx} \right) \right)$. For this reason this representation is called an *image representation* of \mathfrak{B} .

Theorem 3.2 (see [11]). *Let $\mathfrak{B} \in \mathcal{L}_n^w$. Then \mathfrak{B} admits an image representation if and only if it is controllable.*

In this paper, we will assume that the plant is controllable and has an image representation $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dx} \right) \right)$.

Remark 3.3. The elements of \mathfrak{B} of compact support form a subspace of \mathfrak{B} that contains, in general, less information than \mathfrak{B} . However, it was proven in [22, Lemma 2.1] that the compactly supported elements of a controllable behavior are dense in it.

Next, we review the property of observability of nD systems. This property is associated with a given partitioning of the system variables into two disjoint subsets; elements of the first set of variables are interpreted as observed variables and elements of the second as 'to be deduced' variables.

Definition 3.4. Let $\mathfrak{B} \in \mathcal{L}_n^w$ with variable w , and let $w = (w_1, w_2)$ be a partitioning of w . Then w_2 it is said to be observable from w_1 in \mathfrak{B} if given any two trajectories $(w'_1, w'_2), (w''_1, w''_2) \in \mathfrak{B}$ we have that $w'_1 = w''_1$ implies $w'_2 = w''_2$.

Thus, observability is an intrinsic property of the behavior after a partition of the variable w is given. Although we can partition the set of variables in many ways, a natural issue when looking at a latent variable representation of the behavior is to ask whether the latent variables are observable from the manifest variables. If this is the case we call the latent variable representation *observable*. For controllable 1D behaviors it can be shown that there always exists an *observable* image representation. This is not true for nD behaviors (see [13]).

Remark 3.5. Suppose $w = M \left(\frac{d}{dx} \right) \ell$ is an observable image representation of \mathfrak{B} . It can be shown that there exist a polynomial matrix $M^\dagger(\xi) \in \mathbb{R}^{l \times w}[\xi]$ such that $M^\dagger M = I_{l \times l}$, with $I_{l \times l}$ the $l \times l$ the identity matrix. Therefore from $w = M \left(\frac{d}{dx} \right) \ell$,

$$M^\dagger \left(\frac{d}{dx} \right) w = \ell.$$

For more details see [2, Theorem 88]. Thus one has that $w \in \mathfrak{B}$ has compact support if and only if the corresponding $\ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ has compact support.

Unfortunately, for many models of physical systems an observable image representation does not exist. An important instance of this phenomenon is the controllable behavior described by Maxwell equations in free space, see [13, Section 7]. Yet, for every controllable behavior there exists a possibly non-observable image representations, with the property that for every w of compact support there exists an underlying latent variable trajectory ℓ of compact support. This follows from the fact that the set of smooth functions of compact support is a flat $\mathbb{R}[\xi_1, \dots, \xi_n]$ -module (is in fact *faithfully flat*), see [23, Proposition 2.1].

Lemma 3.6. *Let $\mathfrak{B} \in \mathcal{L}_n^w$ be a controllable behavior. There exists an image representation $\mathfrak{B} = \text{im} \left(\tilde{M} \left(\frac{d}{dx} \right) \right)$ of \mathfrak{B} with the property that for all $w \in \mathfrak{B}$ of compact support there exists a $\ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ of compact support such that $w = \tilde{M} \left(\frac{d}{dx} \right) \ell$.*

Definition 3.7. Let $\mathfrak{B} \in \mathcal{L}_n^w$ be a controllable behavior. An image representation $\mathfrak{B} = \text{im} \left(\tilde{M} \left(\frac{d}{dx} \right) \right)$ of \mathfrak{B} with the property described in Lemma 3.6 is called a *faithful image representation* of \mathfrak{B} .

The row span (over $\mathbb{R}[\xi_1, \dots, \xi_n]$) of any matrix that induces a kernel representation of \mathfrak{B} is equal to \mathfrak{B}^\perp . However, the polynomial matrices that induce image representations of \mathfrak{B} do not necessarily have the same column (nor row) span (over $\mathbb{R}[\xi_1, \dots, \xi_n]$). In the case we have a non-faithful image representation of \mathfrak{B} , say $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dx} \right) \right)$, we can construct a polynomial matrix that induces a faithful representation in the following way: consider the behavior $\hat{\mathfrak{B}} = \ker \left(M^\top \left(\frac{d}{dx} \right) \right)$. Then, the transpose of any polynomial matrix that induces a kernel representation of the controllable part of $\hat{\mathfrak{B}}$ (defined as the largest controllable sub-behavior of $\hat{\mathfrak{B}}$) induces a faithful image representation for \mathfrak{B} . Several algorithms exist for computing the controllable part of an nD behavior, see for instance [31, pp. 142–143]. For a more detailed account on the different image representations of a controllable behavior \mathfrak{B} we refer to [23]. For an interesting parametrization of non-controllable systems, see [16].

4. Quadratic differential forms

In many modeling and control problems for linear systems, quadratic functionals of the system variables and their derivatives are involved, for example, in linear-quadratic optimal control, \mathcal{H}_∞ -control ([24]) or in the stability analysis of systems and application of higher order Lyapunov functions ([5]). As we shall see, $2n$ -variable polynomial matrices are a proper mathematical tool to express these quadratic functionals, as already shown, for instance, in [29,18,25] for the one-dimensional case and in [4,8,13] for the multidimensional case. In this section we will also briefly discuss the notion of dissipative system which will be a major tool in the rest of the paper.

A quadratic differential form (QDF) is a quadratic form in the components of a function $w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ and its higher order derivatives. In order to simplify the notation, we denote the vector $\mathbf{x} := (x_1, \dots, x_n)$, the multi-indices $\mathbf{k} := (k_1, \dots, k_n)$ and $\mathbf{l} := (l_1, \dots, l_n)$, and use the notation $\zeta := (\zeta_1, \dots, \zeta_n)$, $\xi := (\xi_1, \dots, \xi_n)$ and $\eta := (\eta_1, \dots, \eta_n)$. Let $\mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$ denote the set of real polynomial $w_1 \times w_2$ matrices in the $2n$ indeterminates ζ and η ; that is, an element of $\mathbb{R}^{w_1 \times w_2}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k}, \mathbf{l}} \Phi_{\mathbf{k}, \mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}},$$

where $\Phi_{\mathbf{k}, \mathbf{l}} \in \mathbb{R}^{w_1 \times w_2}$; the sum ranges over the non-negative multi-indices \mathbf{k} and \mathbf{l} , and is assumed to be finite. Such $2n$ -variable polynomial matrix induces a *bilinear differential form* L_Φ

$$L_\Phi : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w_1}) \times \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w_2}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

$$L_\Phi(v, w) := \sum_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}} \right)^\top \Phi_{\mathbf{k}, \mathbf{l}} \frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}$$

where the \mathbf{k} th derivative operator $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}}$ is defined as $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} := \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$ (similarly for $\frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}}$). Note that ζ corresponds to differentiation of terms on the left and η refers to the terms on the right.

The $2n$ -variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. Note that the former condition is equivalent to $L_\Phi(w_1, w_2) = L_\Phi(w_2, w_1)$ for all w_1, w_2 . We denote the subset of symmetric elements of $\mathbb{R}^{w \times w}[\zeta, \eta]$ by $\mathbb{R}_S^{w \times w}[\zeta, \eta]$. If Φ is symmetric then it induces also a quadratic functional

$$Q_\Phi : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

$$Q_\Phi(w) := L_\Phi(w, w).$$

We will call Q_Φ the *quadratic differential form* associated with Φ .

For a given symmetric $\Phi(\zeta, \eta) \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$ a unique symmetric matrix $\text{mat}(\Phi)$ can be defined. This matrix $\text{mat}(\Phi)$ has as block entries the coefficients $\Phi_{\mathbf{k}, \mathbf{l}}$, with \mathbf{k} and \mathbf{l} ordered according to a given ordering, here taken to be the anti-lexicographic ordering. We will explain this now in more detail for $n = 2$. Let

$$\Phi(\zeta, \eta) = \sum_{k_1, l_1=0}^{N_1} \sum_{k_2, l_2=0}^{N_2} \Phi_{(k_1, k_2)(l_1, l_2)} \zeta_1^{k_1} \zeta_2^{k_2} \eta_1^{l_1} \eta_2^{l_2}.$$

Define $V(\zeta_1) = (I, \zeta_1 I, \dots, \zeta_1^{N_1} I)$, with I the $w \times w$ identity matrix. Then $\Phi(\zeta, \eta)$ can be written as

$$\Phi(\zeta, \eta) = (V(\zeta_1), \zeta_2 V(\zeta_1), \dots, \zeta_2^{N_2} V(\zeta_1)) \begin{pmatrix} \Phi^{(0,0)} & \Phi^{(0,1)} & \dots & \Phi^{(0,N_2)} \\ \Phi^{(1,0)} & \Phi^{(1,1)} & \dots & \Phi^{(1,N_2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{(N_2,0)} & \Phi^{(N_2,1)} & \dots & \Phi^{(N_2,N_2)} \end{pmatrix} \begin{pmatrix} V(\eta_1) \\ \eta V(\eta_1) \\ \vdots \\ \eta^{N_2} V(\eta_1) \end{pmatrix}, \quad (5)$$

where the $(N_1 + 1)w \times (N_1 + 1)w$ matrices $\Phi^{(i,j)}$ are defined by

$$\Phi^{(i,j)} := \begin{pmatrix} \Phi_{(0,i)(0,j)} & \Phi_{(0,i)(1,j)} & \dots & \Phi_{(0,i)(N_1,j)} \\ \Phi_{(1,i)(0,j)} & \Phi_{(1,i)(1,j)} & \dots & \Phi_{(1,i)(N_1,j)} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{(N_1,i)(0,j)} & \Phi_{(N_1,i)(1,j)} & \dots & \Phi_{(N_1,i)(N_1,j)} \end{pmatrix}.$$

Note that $\Phi^{(i,j)}$ has as blocks entries the coefficient matrices $\Phi_{(k_1,i)(l_1,j)}$ of $\Phi(\zeta, \eta)$ with multi-indices (k_1, i) and (l_1, j) ordered in the given ordering. The matrix in the middle of (5) is the coefficient matrix $\text{mat}(\Phi)$ of $\Phi(\zeta, \eta)$. We will denote $(V(\zeta_1), \zeta_2 V(\zeta_1), \dots, \zeta_2^{N_2} V(\zeta_1))$ by $V_2(\zeta_1, \zeta_2)$. Observe that

$$V_2(\zeta_1, \zeta_2) = (I, \zeta_1 I, \dots, \zeta_1^{N_1} I; \zeta_2 I, \zeta_2 \zeta_1 I, \dots, \zeta_2 \zeta_1^{N_1} I; \dots; \zeta_2^{N_2} I, \zeta_2^{N_2} \zeta_1 I, \dots, \zeta_2^{N_2} \zeta_1^{N_1} I)$$

and that the monomials $\zeta_1^{k_1} \zeta_2^{k_2}$ appearing in this matrix from the left to the right are indeed ordered according to the given ordering. Of course, in the same way we can define the coefficient matrix for any n .

We will also consider vectors $\Psi \in (\mathbb{R}_S^{w \times w}[\zeta, \eta])^n$, i.e., n -tuples of symmetric $2n$ -variable polynomial matrices $\Psi = (\Psi_1, \dots, \Psi_n)$ with $\Psi_i \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$. In the same way that Φ induces a QDF, Ψ induces a vector of quadratic differential forms (VQDF), defined as

$$Q_\Psi : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \longrightarrow (\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}))^n \\ Q_\Psi(w) := (Q_{\Psi_1}(w), Q_{\Psi_2}(w), \dots, Q_{\Psi_n}(w)).$$

Given a VQDF Q_Ψ as above, its divergence is defined as the QDF

$$(\text{div } Q_\Psi)(w) := \frac{\partial}{\partial x_1} Q_{\Psi_1}(w) + \dots + \frac{\partial}{\partial x_n} Q_{\Psi_n}(w) \quad (6)$$

for $w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$.

It is easy to verify that the $2n$ -variable polynomial matrix associated with the divergence of a VQDF Q_Ψ is given by the $2n$ -variable polynomial matrix $\dot{\Psi}$ defined by

$$\dot{\Psi}(\zeta, \eta) := (\zeta_1 + \eta_1) \Psi_1(\zeta, \eta) + \dots + (\zeta_n + \eta_n) \Psi_n(\zeta, \eta).$$

We refer to [4,13,29] for a deeper study of QDF's.

The useful notion of *dissipativity* lies at the root of many stability results and will play an important role in the following.

Definition 4.1. Let $\Phi \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$. Let $\mathfrak{B} \in \mathcal{L}_n^w$ be a controllable behavior. \mathfrak{B} is said to be Q_Φ -dissipative if

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0 \text{ for all } w \in \mathfrak{B} \text{ with compact support.}$$

The functional $Q_\Phi(w)$ is often interpreted as the rate of supply of some physical quantity (for example, energy) which flows into the system if the system produces the signal $w(\mathbf{x})$ (whence positive when the system absorbs supply). Thus $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$ is the total net energy delivered to the system by taking it through the trajectory w , and dissipativity states that the system absorbs energy (in space and time) during any trajectory in \mathfrak{B} that starts and ends with the system at rest.

Definition 4.2. Let $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{d\mathbf{x}} \right) \right)$ be a controllable behavior and Φ a QDF. The VQDF Q_Ψ is called a storage function for \mathfrak{B} with respect to Φ if

$$\text{div} Q_\Psi(\ell) \leq Q_\Phi(w)$$

for all $w \in \mathfrak{B}$ and $w = M \left(\frac{d}{d\mathbf{x}} \right) \ell$.

If Q_Φ is thought as the rate at which supply is delivered to the system, then the storage function Q_Ψ can be interpreted as total supply stored inside the system.

If $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ is Q_Φ -dissipative, then we call the QDF Q_Φ *average non-negative*:

Definition 4.3. Let $\Phi \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$. Then Q_Φ is said to be average non-negative if

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0 \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \text{ of compact support.}$$

Using Lemma 3.6, for a given controllable behavior \mathfrak{B} and a $2n$ -variable polynomial matrix $\Phi(\zeta, \eta) \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$ we can express Q_Φ -dissipativity in terms of average non-negativity of an auxiliary $2n$ -variable polynomial matrix $\Phi'(\zeta, \eta)$ associated with Φ and an appropriate image representation of \mathfrak{B} . In general, if $w = M \left(\frac{d}{d\mathbf{x}} \right) \ell$ is an image representation of \mathfrak{B} , define Φ' by

$$\Phi'(\zeta, \eta) := M^\top(\zeta) \Phi(\zeta, \eta) M(\eta). \quad (7)$$

Denote the coefficients of the $2n$ -variable polynomial matrix Φ' by $\Phi'_{\mathbf{k}, \mathbf{l}}$. Then, if w_1, ℓ_1 and w_2, ℓ_2 are related by $w_1 = M \left(\frac{d}{d\mathbf{x}} \right) \ell_1$ and $w_2 = M \left(\frac{d}{d\mathbf{x}} \right) \ell_2$, we have

$$\begin{aligned} L_{\Phi'}(\ell_1, \ell_2) &= \sum_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{k}} \ell_1}{d\mathbf{x}^{\mathbf{k}}} \right)^\top \Phi'_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{l}} \ell_2}{d\mathbf{x}^{\mathbf{l}}} \right) \\ &= \sum_{\mathbf{k}, \mathbf{l}} \left(M \left(\frac{d}{d\mathbf{x}} \right) \frac{d^{\mathbf{k}} \ell_1}{d\mathbf{x}^{\mathbf{k}}} \right)^\top \Phi_{\mathbf{k}, \mathbf{l}} M \left(\frac{d}{d\mathbf{x}} \right) \left(\frac{d^{\mathbf{l}} \ell_2}{d\mathbf{x}^{\mathbf{l}}} \right) \\ &= \sum_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} M \left(\frac{d}{d\mathbf{x}} \right) \ell_1 \right)^\top \Phi_{\mathbf{k}, \mathbf{l}} \frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}} M \left(\frac{d}{d\mathbf{x}} \right) \ell_2 \\ &= \sum_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{k}} w_1}{d\mathbf{x}^{\mathbf{k}}} \right)^\top \Phi_{\mathbf{k}, \mathbf{l}} \frac{d^{\mathbf{l}} w_2}{d\mathbf{x}^{\mathbf{l}}} \\ &= L_\Phi(w_1, w_2). \end{aligned}$$

Then, by taking a faithful image representation of \mathfrak{B} , we have:

Proposition 4.4. Let $\mathfrak{B} \in \mathcal{L}_n^W$ be a controllable behavior. Let $\mathfrak{B} = \text{im} \left(\tilde{M} \left(\frac{d}{dx} \right) \right)$ be a faithful image representation of \mathfrak{B} and define

$$\Phi'(\zeta, \eta) := \tilde{M}^\top(\zeta) \Phi(\zeta, \eta) \tilde{M}(\eta).$$

Then \mathfrak{B} is Q_Φ -dissipative if and only if $Q_{\Phi'}$ is average non-negative.

5. Stationary and local minimum trajectories

In this section we will introduce the notions of stationary and local minimum trajectories, where the variation functions are taken as smooth functions with compact support. We will characterize the space of stationary trajectories of a behavior and show that this space coincides with the set of local minimum trajectories if the system is dissipative, and is empty otherwise. Moreover, we indicate how to check whether a system is dissipative or not.

Definition 5.1. Let $\mathfrak{B} \in \mathcal{L}_n^W$ be controllable and let $\Phi \in \mathbb{R}_S^{W \times W}[\zeta, \eta]$. The trajectory $w \in \mathfrak{B}$ is called *stationary with respect to* $\int_{\mathbb{R}^n} Q_\Phi(\cdot) d\mathbf{x}$ (or: Φ -stationary) if for all $\Delta \in \mathfrak{B}$ of compact support we have

$$\int_{\mathbb{R}^n} L_\Phi(\Delta, w) d\mathbf{x} = 0.$$

It is easy to check that $w \in \mathfrak{B}$ is Φ -stationary if and only if for all $\Delta \in \mathfrak{B}$ of compact support we have

$$\int_{\mathbb{R}^n} Q_\Phi(w + \Delta) - Q_\Phi(w) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(\Delta) d\mathbf{x}.$$

Theorem 5.2. Let $\Phi \in \mathbb{R}_S^{W \times W}[\zeta, \eta]$, $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dx} \right) \right)$, and define $\Phi'(\zeta, \eta) := M^\top(\zeta) \Phi(\zeta, \eta) M(\eta)$. Then $w \in \mathfrak{B}$ is Φ -stationary if and only if $w = M \left(\frac{d}{dx} \right) \ell$, with $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ a solution of

$$\Phi' \left(-\frac{d}{dx}, \frac{d}{dx} \right) \ell = 0. \quad (8)$$

Proof. For any pair of functions $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^W)$, with w_1 of compact support, integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^n} L_\Phi(w_1, w_2) d\mathbf{x} &= \int_{\mathbb{R}^n} \sum_{\mathbf{k}, \mathbf{l}} \left(\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} w_1 \right)^\top \Phi_{\mathbf{k}, \mathbf{l}} \frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}} w_2 d\mathbf{x} \\ &= \int_{\mathbb{R}^n} w_1^\top \sum_{\mathbf{k}, \mathbf{l}} \Phi_{\mathbf{k}, \mathbf{l}} (-1)^{\mathbf{k}} \frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} \frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}} w_2 d\mathbf{x} \\ &= \int_{\mathbb{R}^n} w_1^\top \Phi \left(-\frac{d}{dx}, \frac{d}{dx} \right) w_2 d\mathbf{x}. \end{aligned}$$

Now suppose that $w = M \left(\frac{d}{dx} \right) \ell$ is Φ -stationary, i.e.,

$$\int_{\mathbb{R}^n} \Delta^\top \Phi \left(-\frac{d}{dx}, \frac{d}{dx} \right) M \left(\frac{d}{dx} \right) \ell d\mathbf{x} = 0$$

for all $\Delta \in \mathfrak{B}$ of compact support. Then we have

$$\int_{\mathbb{R}^n} \left(M \left(\frac{d}{dx} \right) \tilde{\Delta} \right)^\top \Phi \left(-\frac{d}{dx}, \frac{d}{dx} \right) M \left(\frac{d}{dx} \right) \ell d\mathbf{x} = 0$$

for all $\tilde{\Delta} \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ of compact support. Integrating by parts we obtain

$$\int_{\mathbb{R}^n} \tilde{\Delta}^\top M^\top \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) \Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) M \left(\frac{d}{d\mathbf{x}} \right) \ell \, d\mathbf{x} = 0$$

for all $\tilde{\Delta} \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ of compact support, which implies

$$M^\top \left(-\frac{d}{d\mathbf{x}} \right) \Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) M \left(\frac{d}{d\mathbf{x}} \right) \ell = 0,$$

equivalently, $\Phi' \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) \ell = 0$.

In order to prove the converse, let $w = \tilde{M} \left(\frac{d}{d\mathbf{x}} \right) \ell$ be a faithful image representation of \mathfrak{B} . Let w be such that $w = M \left(\frac{d}{d\mathbf{x}} \right) \ell$ with $\Phi' \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) \ell = 0$. Then we have

$$\Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) M \left(\frac{d}{d\mathbf{x}} \right) \ell \in \ker \left(M^\top \left(-\frac{d}{d\mathbf{x}} \right) \right).$$

Furthermore, from $\text{im} \left(M \left(\frac{d}{d\mathbf{x}} \right) \right) = \mathfrak{B} = \text{im} \left(\tilde{M} \left(\frac{d}{d\mathbf{x}} \right) \right)$ it follows that $\ker \left(M^\top \left(-\frac{d}{d\mathbf{x}} \right) \right) = \ker \left(\tilde{M}^\top \left(-\frac{d}{d\mathbf{x}} \right) \right)$. Hence, $\tilde{M}^\top \left(-\frac{d}{d\mathbf{x}} \right) \Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) M \left(\frac{d}{d\mathbf{x}} \right) \ell = 0$. Now let $\Delta \in \mathfrak{B}$ be of compact support and let $\tilde{\Delta}$ of compact support be such that $\Delta = \tilde{M} \left(\frac{d}{d\mathbf{x}} \right) \tilde{\Delta}$. Again integrating by parts we then obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \Delta^\top \Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) w \, d\mathbf{x} \\ &= \int_{\mathbb{R}^n} \tilde{\Delta}^\top \tilde{M}^\top \left(-\frac{d}{d\mathbf{x}} \right) \Phi \left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}} \right) M \left(\frac{d}{d\mathbf{x}} \right) \ell \, d\mathbf{x} = 0 \end{aligned}$$

and therefore w is Φ -stationary. \square

See [19] for complementary material on stationary trajectories.

Example 5.3. Let $\mathfrak{B} = \text{im} \left(M \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \right) \subset \mathfrak{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be the behavior represented by

$$M(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1^2 \xi_2 \\ 0 & 1 + \xi_3 \end{pmatrix}.$$

Consider the QDF $Q_\Phi(w_1, w_2, w_3) := 2w_1 \frac{\partial}{\partial x_1} w_3 + 3w_2^2$. This QDF is associated with the symmetric 6 variable polynomial matrix

$$\Phi(\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3) = \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 3 & 0 \\ \eta_1 & 0 & 0 \end{pmatrix}.$$

The sub-behavior of stationary trajectories of \mathfrak{B} with respect to $\int_{\mathbb{R}^n} Q_\Phi(\cdot) d\mathbf{x}$ equals $M \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \ker \left(S \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \right)$, with $S(\xi) := M^\top(-\xi) \Phi(-\xi, \xi) M(\xi)$ computed as

$$S(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} 0 & \xi_1^2(1 + \xi_3) \\ \xi_1^2(1 - \xi_3) & 3\xi_1^4 \xi_2^2 \end{pmatrix}.$$

Example 5.4. In this example we consider the behavior $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^3)$ together with the QDF

$$Q_\Phi(w) = \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 - \frac{1}{2} \left(\left(\frac{\partial w}{\partial x_2} \right)^2 + \left(\frac{\partial w}{\partial x_3} \right)^2 \right).$$

which is associated with the 6-variable polynomial

$$\Phi(\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3) := \frac{1}{2}\zeta_1\eta_1 - \frac{1}{2}(\zeta_2\eta_2 + \zeta_3\eta_3).$$

Obviously, $\mathfrak{B} = \text{im}(I)$, with I the 3×3 identity matrix. The sub-behavior of Φ -stationary trajectories is then computed as $\ker\left(S\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)\right)$, with $S(\xi) = \Phi(-\xi, \xi) = -\frac{1}{2}\xi_1^2 + \frac{1}{2}(\xi_2^2 + \xi_3^2)$. Thus the sub-behavior of Φ -stationary trajectories is represented by

$$\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_3^2} = 0.$$

By interpreting x_1 as time t , and (x_2, x_3) as position, this equation describes the transversal displacement $w(x_1, x_2, x_3)$ from equilibrium at time x_1 of the point (x_2, x_3) of a homogeneous flexible sheet (membrane). The QDF $Q_\Phi(w)$ represents the Lagrangian (the difference between the kinetic and potential energy).

Next we examine when and in what sense a Φ -stationary trajectory is a local minimum.

Definition 5.5. Let $\mathfrak{B} \in \mathcal{L}_n^w$ be controllable and let $\Phi \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$. A trajectory $w \in \mathfrak{B}$ is called a *local minimum* or *optimal* for $\int_{\mathbb{R}^n} Q_\Phi(\cdot) d\mathbf{x}$ with respect to compact support variations if

$$\int_{\mathbb{R}^n} Q_\Phi(w + \Delta) - Q_\Phi(w) d\mathbf{x} \geq 0,$$

for all $\Delta \in \mathfrak{B}$ with compact support.

Thus, a trajectory in \mathfrak{B} is a local minimum if it cannot be “improved” by adding a compactly supported trajectory to it. This formulation will, in fact, lead to the existence of many locally minimal trajectories. In the case $\mathfrak{B} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ this definition of optimal trajectories coincides with the one given in [12,19].

The following theorem gives an explicit condition under which stationary trajectories are local minima:

Theorem 5.6. Let $\mathfrak{B} \in \mathcal{L}_n^w$ be controllable and let $\Phi \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$. If \mathfrak{B} is Q_Φ -dissipative then the set of locally minimal trajectories is equal to the set of Φ -stationary trajectories. If \mathfrak{B} is not Q_Φ -dissipative, then the set of locally minimal trajectories is empty.

Proof. For any w and any Δ of compact support we have

$$\int_{\mathbb{R}^n} Q_\Phi(w + \Delta) - Q_\Phi(w) d\mathbf{x} = 2 \int_{\mathbb{R}^n} L_\Phi(w, \Delta) d\mathbf{x} + \int_{\mathbb{R}^n} Q_\Phi(\Delta) d\mathbf{x}.$$

Suppose \mathfrak{B} is Q_Φ -dissipative. Let $w \in \mathfrak{B}$ be a local minimum. One needs to prove that $\int_{\mathbb{R}^n} \Delta^\top \Phi\left(-\frac{d}{d\mathbf{x}}, \frac{d}{d\mathbf{x}}\right) w d\mathbf{x} = 0$ for all $\Delta \in \mathfrak{B}$ of compact support. Assume there exists a $\bar{\Delta} \in \mathfrak{B}$ of compact support such that $\int_{\mathbb{R}^n} L_\Phi(w, \bar{\Delta}) d\mathbf{x} \neq 0$. Since $\int_{\mathbb{R}^n} L_\Phi(w, \bar{\Delta}) d\mathbf{x}$ and $\int_{\mathbb{R}^n} Q_\Phi(\bar{\Delta}) d\mathbf{x}$ are fixed numbers and $\lambda \bar{\Delta}$ is again a compact support trajectory for all $\lambda \in \mathbb{R}$, there clearly exists a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} & 2 \int_{\mathbb{R}^n} L_\Phi(w, \lambda \bar{\Delta}) d\mathbf{x} + \int_{\mathbb{R}^n} Q_\Phi(\lambda \bar{\Delta}) d\mathbf{x} \\ &= 2\lambda \int_{\mathbb{R}^n} L_\Phi(w, \bar{\Delta}) d\mathbf{x} + \lambda^2 \int_{\mathbb{R}^n} Q_\Phi(\bar{\Delta}) d\mathbf{x} < 0. \end{aligned}$$

This contradicts the assumption that w is a local minimum and therefore we have $\int_{\mathbb{R}^n} L_\Phi(w, \Delta) d\mathbf{x} = 0$ for all $\Delta \in \mathfrak{B}$ of compact support, which proves that w is Φ -stationary. Conversely, if \mathfrak{B} is Q_Φ -dissipative then clearly every Φ -stationary trajectory is a local minimum.

Now assume that \mathfrak{B} is not Q_Φ -dissipative. Then there exists $\bar{\Delta} \in \mathfrak{B}$ of compact support such that $\int_{\mathbb{R}^n} Q_\Phi(\bar{\Delta}) d\mathbf{x} < 0$. Then for any $w \in \mathfrak{B}$ there exists a suitable $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} & 2 \int_{\mathbb{R}^n} L_\Phi(w, \lambda \bar{\Delta}) d\mathbf{x} + \int_{\mathbb{R}^n} Q_\Phi(\lambda \bar{\Delta}) d\mathbf{x} \\ & = 2\lambda \int_{\mathbb{R}^n} L_\Phi(w, \bar{\Delta}) d\mathbf{x} + \lambda^2 \int_{\mathbb{R}^n} Q_\Phi(\bar{\Delta}) d\mathbf{x} < 0, \end{aligned}$$

while $\lambda \bar{\Delta} \in \mathfrak{B}$ and has compact support. This proves that w cannot be a local minimum and therefore the set of locally minimal trajectories is empty. \square

As a consequence of the previous theorem, we have that the existence of (non-trivial) locally minimal trajectories depends crucially on the whether the behavior \mathfrak{B} is Q_Φ -dissipative. Next we make use of faithful representations of \mathfrak{B} in order to provide a constructive method to check whether or not \mathfrak{B} is Q_Φ -dissipative.

Theorem 5.7. *Let Q_Φ be a QDF, $\mathfrak{B} \in \mathcal{L}_n^w$ a controllable behavior and $\mathfrak{B} = \text{im} \left(\tilde{M} \left(\frac{d}{dx} \right) \right)$ a faithful image representation of \mathfrak{B} . Then, \mathfrak{B} is Q_Φ -dissipative if and only if there exists a VQDF $\Psi(\zeta, \eta) = (\Psi_1(\zeta, \eta), \dots, \Psi_n(\zeta, \eta))$ with $\Psi_i \in \mathbb{R}_S^{w \times w}[\zeta, \eta]$, $i = 1, \dots, n$, and a polynomial $p \in \mathbb{R}[\xi]$ such that*

$$\text{mat}(\hat{\Phi} - \dot{\Psi}) \geq 0, \quad (9)$$

with $\hat{\Phi}(\zeta, \eta) := \tilde{M}(\zeta)^\top \Phi(\zeta, \eta) \tilde{M}(\eta) p(\zeta) p(\eta)$.

Proof. By Proposition 4.4, \mathfrak{B} is Q_Φ -dissipative amounts to saying that $Q_{\Phi'}$, with $\Phi'(\zeta, \eta) = \tilde{M}(\zeta)^\top \Phi(\zeta, \eta) \tilde{M}(\eta)$, is average non-negative. Moreover, [13, Proposition 11] implies that $Q_{\Phi'}$ is average non-negative if and only if $\Phi'(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Finally, the equivalence of equation (9) and the condition $\Phi'(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$ was proved in [8, Theorem 4.2]. \square

Condition (9) can be restated as $p(\zeta) \tilde{M}^\top(\zeta) \Phi(\zeta, \eta) \tilde{M}(\eta) p(\eta) - \dot{\Psi}(\zeta, \eta) = D^\top(\zeta) D(\eta)$ for some polynomial matrix D of suitable size, or equivalently, $\text{div}(Q_\Psi(\ell)) - Q_\Phi \left(\tilde{M} \left(\frac{d}{dx} \right) p \left(\frac{d}{dx} \right) \ell \right) = \left\| D \left(\frac{d}{dx} \right) \ell \right\|^2$ for all $\ell \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$. This says that the nD system behavior with image representation $w = \tilde{M} \left(\frac{d}{dx} \right) p \left(\frac{d}{dx} \right) \ell$ is Q_Φ -dissipative and has storage function Q_Ψ , see [13] for more details.

The central issue here is that the solvability of the inequality (9) can be checked computationally. The unknowns in (9) are the entries of $\text{mat}(\Psi_i)$, $i = 1, 2, \dots, n$, and the coefficients of p . Of course, the coefficients of p appear quadratically in the inequality, and therefore (9) is not a linear matrix inequality. However, there exist several algorithms to check this inequality. For instance, it can be reduced to a linear eigenvalue problem or to a problem of checking whether a semi-algebraic set is empty, a problem that can be effectively treated using semidefinite programming [10], see [8] for details on these algorithms. Hence, Theorem 5.7 provides a criterion to efficiently check whether the set of locally minimal trajectories of \mathfrak{B} is empty or coincides with the set of stationary trajectories of \mathfrak{B} .

6. Regular implementation of the stationary trajectories

In the behavioral framework, control is based on interconnection of systems. While a plant behavior $\mathfrak{B} \in \mathcal{L}_n^w$ consists of all trajectories satisfying a set of differential equations, one would like to restrict this space of trajectories to a desired subsystem, $\mathcal{K} \subset \mathfrak{B}$. This restriction can be effected by increasing the number of equations that the variables of the plant have to satisfy. These additional laws themselves define a new system, called the controller (denoted by \mathcal{C}). The interconnection of the two systems (the plant and the controller) results in the controlled behavior \mathcal{K} . After interconnection, the variables have to satisfy the laws of both \mathfrak{B} and \mathcal{C} . The *interconnection* of \mathfrak{B} and \mathcal{C} is defined as the system with

behavior $\mathfrak{B} \cap \mathcal{C}$. Note that $\mathfrak{B} \cap \mathcal{C}$ is again an element of \mathcal{L}_n^w . If, for a given $\mathcal{K} \in \mathcal{L}_n^w$, we have $\mathcal{K} = \mathfrak{B} \cap \mathcal{C}$ then we say that the controller \mathcal{C} implements \mathcal{K} .

Whereas in the classical LQ problem a feedback controller is sought, in the behavioral context we look at the more fundamental concept of control by *regular* interconnection, see [28,32,21]. The interconnection of \mathfrak{B} and \mathcal{C} is called *regular*, if

$$p(\mathfrak{B} \cap \mathcal{C}) = p(\mathfrak{B}) + p(\mathcal{C}),$$

where p is equal to the rank of the polynomial matrix in any kernel representation of \mathfrak{B} . Equivalently, the interconnection of \mathfrak{B} and \mathcal{C} is regular if and only if $\mathfrak{B} + \mathcal{C} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$, (see [21, Lemma 3.3]). Regular interconnection expresses the idea of “restricting what is not restricted”. In a regular interconnection, the controller imposes new restrictions on the plant; it does not reimpose restrictions that are already present. In this sense, the controller in a regular interconnection has no redundancy. Moreover, regular interconnection is both necessary and sufficient for the existence of a feedback control structure (see [20]). A given behavior $\mathcal{K} \in \mathcal{L}_n^w$ is called *regularly implementable with respect to* \mathfrak{B} if there exists a $\mathcal{C} \in \mathcal{L}_n^w$ such that $\mathcal{K} = \mathfrak{B} \cap \mathcal{C}$, and the interconnection is regular. In that case we say that \mathcal{K} is *regularly implemented* by the controller \mathcal{C} .

In the following theorem we provide conditions for the existence of a controller that implements the sub-behavior of stationary trajectories through a regular interconnection. Also, an explicit representation of such controller is given.

Theorem 6.1. Let $\mathfrak{B} \in \mathcal{L}_n^w$ be controllable and let $\Phi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$. Let $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dx} \right) \right)$ be an image representation of \mathfrak{B} . Let $\Phi'(\zeta, \eta) := M^\top(\zeta) \Phi(\zeta, \eta) M(\eta)$ and assume that $\det(\Phi'(-\xi, \xi)) \neq 0$. Then the sub-behavior of stationary trajectories is regularly implementable and is regularly implemented by the controller $\mathcal{C} := \ker \left(C \left(\frac{d}{dx} \right) \right)$ with

$$C(\xi) := M^\top(-\xi) \Phi(-\xi, \xi). \quad (10)$$

Proof. The claim that $\mathfrak{B} \cap \mathcal{C}$ is equal to the sub-behavior of stationary trajectories follows immediately from Theorem 5.2. To see that the interconnection is regular we need to check that $\mathfrak{B} + \mathcal{C} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$. Now, a kernel representation of $\mathfrak{B} + \mathcal{C}$ is obtained as follows. Let $\mathfrak{B} = \ker \left(R \left(\frac{d}{dx} \right) \right)$ be a kernel representation of \mathfrak{B} . Consider the polynomial matrix $\begin{pmatrix} R \\ C \end{pmatrix}$ and let $(N \quad L)$ be a polynomial matrix such that

$$\ker \left(N \left(\frac{d}{dx} \right) \quad L \left(\frac{d}{dx} \right) \right) = \text{im} \left(R \left(\frac{d}{dx} \right) \right).$$

Then obviously $NR = -LC$ and according to ([21], Lemma 2.14), $\mathfrak{B} + \mathcal{C} = \ker \left(N \left(\frac{d}{dx} \right) R \left(\frac{d}{dx} \right) \right)$. In our case we have $C(\xi) = M^\top(-\xi) \Phi(-\xi, \xi)$ so therefore $N(\xi)R(\xi) + L(-\xi)M^\top(-\xi) \Phi(-\xi, \xi) = 0$. Hence for every $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^l)$ we get

$$L \left(\frac{d}{dx} \right) M^\top \left(-\frac{d}{dx} \right) \Phi \left(-\frac{d}{dx}, \frac{d}{dx} \right) M \left(\frac{d}{dx} \right) \ell = 0,$$

which implies $L(\xi)M^\top(-\xi) \Phi(-\xi, \xi)M(\xi) = 0$. Since we have assumed that $\Phi'(-\xi, \xi) = M^\top(-\xi) \Phi(-\xi, \xi)M(\xi)$ is nonsingular, this implies $L(\xi) = 0$. Thus $NR = 0$, implying that $\mathfrak{B} + \mathcal{C} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$. This proves that the interconnection is regular. \square

Example 6.2 (*The damped vibrating string*). Let \mathfrak{B} be the behavior represented by the equation

$$\rho_0 \frac{\partial^2}{\partial t^2} w_1 - T_0 \frac{\partial^2}{\partial x^2} w_1 + \beta \frac{\partial}{\partial t} w_1 = w_2,$$

where ρ_0 stands for the density of the string, T_0 for its tension, β for the friction coefficient, w_1 denotes the position and w_2 the (vertical) force. A faithful image representation of $\mathfrak{B} = \text{im}(\tilde{M})$ is given by the matrix

$$\tilde{M}(\xi_t, \xi_x) = \begin{bmatrix} 1 \\ \rho_0 \xi_t^2 - T_0 \xi_x^2 + \beta \xi_t \end{bmatrix}.$$

The supply rate is $\frac{\partial}{\partial t} w_1 \cdot w_2$, which using a symmetric 4-variable polynomial matrix Φ can be represented as

$$\Phi(\zeta_t, \zeta_x, \eta_t, \eta_x) = \begin{pmatrix} 0 & \frac{1}{2}\zeta_t \\ \frac{1}{2}\eta_t & 0 \end{pmatrix}.$$

Aiming to describe the local minima trajectories of \mathfrak{B} (for $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$ with respect to compact support variations) we first check whether the system is Q_Φ -dissipative. It is easy to verify that

$$\Phi'(\zeta_t, \zeta_x, \eta_t, \zeta_x) = \tilde{M}(\zeta_t, \zeta_x)^\top \Phi(\zeta_t, \zeta_x, \eta_t, \eta_x) \tilde{M}(\eta_t, \eta_x) \quad (11)$$

$$= \frac{1}{2}(\rho_0 \zeta_t^2 \eta_t - T_0 \zeta_x^2 \eta_t + 2\beta \zeta_t \eta_t + \rho \zeta_t \eta_t^2 - T_0 \zeta_t \eta_x^2). \quad (12)$$

Further, it can be easily checked that there exist a vector of QDF $\Psi = (\Psi_1, \Psi_2)$ and a polynomial p such that they satisfy the conditions of Theorem 5.7 and hence \mathfrak{B} is Q_Φ -dissipative. Indeed, we can take $p(\xi) = 1$ and

$$\Psi = \begin{bmatrix} \frac{1}{2}(\rho_0 \zeta_t \eta_t + T_0 \zeta_x \eta_x) & \frac{1}{2}(-T_0 \zeta_t \eta_x - T_0 \eta_t \zeta_x) \end{bmatrix}.$$

$Q_\Psi(\ell)$ is a storage function for \mathfrak{B} with respect to Φ , where $\frac{1}{2}\rho_0 \left(\frac{\partial}{\partial t} \ell\right)^2$ can be interpreted as the kinetic energy, $\frac{1}{2}T_0 \left(\frac{\partial}{\partial x} \ell\right)^2$ as the potential energy and $-\frac{1}{2}T_0 \left(\frac{\partial}{\partial x}\right) \ell \left(\frac{\partial}{\partial t}\right) \ell$ as the flux. From Theorems 5.6 and 6.1 we conclude that a controller that regular implements the set of locally minimal trajectories is given by

$$C(\xi_t, \xi_x) = M^\top(-\xi_t, -\xi_x) \Phi(-\xi_t, -\xi_x, \xi_t, \xi_x) = \begin{bmatrix} 1 & \rho_0 \xi_t^2 - T_0 \xi_x^2 - \beta \xi_t \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{1}{2}\xi_t \\ \frac{1}{2}\xi_t & 0 \end{bmatrix}.$$

Example 6.3. Let \mathfrak{B} and Φ be given as in Example 5.3. Then it is easy to see that a controller that regularly implements the stationary trajectories of \mathfrak{B} with respect to $\int_{\mathbb{R}^n} Q_\Phi(\cdot) d\mathbf{x}$ is represented by $C = \ker \left(C \left(\frac{d}{d\mathbf{x}} \right) \right)$, with

$$C(\xi_1, \xi_2, \xi_3) := \begin{pmatrix} 0 & 0 & \xi_1^2 \\ \xi_1(1 - \xi_3) & 3\xi_1^2 \xi_2 & 0 \end{pmatrix}.$$

The set of locally minimal trajectories (for $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$ with respect to compact support variations) is empty since the behavior \mathfrak{B} is not Q_Φ -dissipative.

7. Conclusions

In this paper we have presented a natural framework in which is possible to treat in great generality LQ problems, where no input/output structure of the systems is displayed, and where no state space representation is assumed, i.e., which completely fits in the behavioral context. The optimal control problem addressed here is based on the space of trajectories which locally minimizes a given cost functional, given by a quadratic differential form, against compact support trajectories. We have provided an effective procedure to determine these optimal trajectories and construct a regular controller that implements them.

We expect that this approach will provide further results. Future research will treat, for instance, situations in which stability is imposed, and where a larger classes of trajectory variations are considered.

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